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Ergodic Theorems for Free Group Actions on von Neumann Algebras

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We extend a recent ergodic theorem of A. Nevo and E. Stein to the non-commutative case. Let ρ be a faithful normal state on the von Neumann algebra A. Let $\{a_i\}_{i=1}^{r}$ generate F_r, the free group on r generators, and let $\{\alpha_i\}_{i=1}^{r}$ be $*$-automorphisms of A which leave ρ invariant. Define ϕ to be the group

is a neutral element), and $|w_n|$ as the number of elements of w_n. Let $\sigma_n = (1/|w_n|) \sum_{a \in w_n} \phi(a)$ and $S_n = (1/n) \sum_{k=0}^{n-1} \sigma_k$. We then show that if x is in A, then $S_n(x)$ converges almost uniformly to an element $\hat{x} \in A$. To prove the above theorem, we prove an ergodic theorem involving completely positive maps, of which the free group situation is a special case. Roughly, if $p_1 \geqslant p_2 \geqslant 0$, $p_1 + p_2 = 1$, σ_n positive maps such that $\sigma_1 \circ \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$ $(n \geqslant 1)$, with $\sigma_0(x) = x$ and $\sigma_1(1) = 1$ then, with a few technical assumptions, we show a convergence result for $\lim_{n \to \infty} \sigma_n(x)$ and show that $\lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$ converges almost uniformly. In the case $p_2 = 0$, σ_1 a $*$-automorphism, our theorems correspond to the non-commutative pointwise ergodic theorem of E. C. Lance. The results partially generalize a result of Kummerer. Our theorems also include results concerning normal operators on a Hilbert space which generalizes work of Guivarc'h. © 1997 Academic Press

1. INTRODUCTION

1.1. *Background*

The classical study of ergodic theory has centered around von Neumann's ergodic theorem and Birkhoff's pointwise ergodic theorem. These theorems involved a single measure preserving transform α and sums of the form $1/n \sum_{k=0}^{n-1} \alpha^k$. Later, extensions of these theorems were made to deal with the case when one had several commuting measure preserving transforms. In 1976, E. C. Lance [L] gave a Birkhoff type theorem for an automorphism acting on a von Neumann algebra.

In 1969, Guivarc'h [G] extended von Neumann's ergodic theorem to the case of several non-commuting unitary operators on a Hilbert space.

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Recently, Amos Nevo and Elias Stein [NS] proved the following Birkhoff type theorem for L^p spaces, $1 \leqslant p < \infty$.

THEOREM [NS]. *Given* (X, β, μ) *a Lebesgue probability space. Let* $\{a_i\}_{i=1}^r$ *generate* F_r, *the free group on* r *generators and let* $\{\alpha_i\}_{i=1}^r$ *be measure preserving automorphisms of* X. *Take* ϕ *to be the group homomorphism from* F_r *to the measure preserving automorphisms of* X *defined on base elements by* $\phi: a_i \mapsto \alpha_i$.

Define w_n *as the set of all reduced words in* F_r *of length* n (*the identity is a neutral element*), *and* $|w_n|$ *as the number of elements of* w_n. (*e.g.*, $w_1 = \{a_1, a_2, ..., a_r, a_1^{-1}, a_2^{-1}, ..., a_r^{-1}\}$, *and* $|w_1| = 2r$). *Define* $\sigma_k = (1/|w_n|) \sum_{a \in w_n} \phi(a)$. *Then, for* $f \in L^p(X)$, $1 \leqslant p < \infty$,

$$n^{-1} \sum_{k=0}^{n-1} \sigma_k(f)$$

converges pointwise almost everywhere and in $L^p(X)$.

Since for the Birkhoff ergodic theorem there exists Lance's extension to von Neumann algebras, it was suggested by Dan Voiculescu that it would be natural to look for a corresponding extension to von Neumann algebras for the theorem of [NS].

1.2. *Results*

1.2.1. *Free Group Actions on a von Neumann Algebra*

The following extends results in [NS].

Let ρ be a faithful normal state on the von Neumann algebra A. Let $\{a_i\}_{i=1}^r$ generate F_r, the free group on r generators, and let $\{\alpha_i\}_{i=1}^r$ be $*$-automorphisms of A which leave ρ invariant. Let ϕ be the group homomorphism from F_r to the *-automorphisms of A defined on base elements by $\phi: a_i \mapsto \alpha_i$.

Define w_n as the set of all reduced words in F_r of length n (the identity is a neutral element), and $|w_n|$ as the number of elements of w_n. (e.g. $w_1 = \{a_1, a_2, ..., a_r, a_1^{-1}, a_2^{-1}, ..., a_r^{-1}\}$, and $|w_1| = 2r$). Note these objects vary with respect to r, the number of generators of the free group. Define the following elements as in [NS].

$$\sigma_n = \frac{1}{|w_n|} \sum_{a \in w_n} \phi(a), \qquad S_n = \frac{1}{n} \sum_{k=0}^{n-1} \sigma_k. \tag{1}$$

We refer to the above as the *Free Group Actions* and the *Free Group Partial Sums* respectively. We then show that if x is in A, then $S_n(x)$ converges almost uniformly to an element $\hat{x} \in A$. We remind the reader that $S_n(x)$ converges to $\hat{x}$ almost uniformly if given $\varepsilon > 0$ then there exists a projection $p \in A$ such that $\rho(I-p) < \varepsilon$ and

$$\lim_{n \to \infty} \|(S_n(x) - \hat{x})p\| = 0.$$

We will use the well known fact that $\sigma_1 \circ \sigma_k = p_1 \sigma_{k+1} + p_2 \sigma_{k-1}$, $k > 1$, where $p_1 = (2r-1)/2r$, and $p_2 = 1/2r$ to prove the above result by proving the results described in 1.2.2 below, which generalize a result of E. C. Lance, and partially generalize a result of Kummerer.

1.2.2. *The General Case*

Let $\sigma_1 : A \to A$ be a normal completely positive map on a von Neumann algebra A, $\sigma_1(1) = 1$, σ_0 the identity map ($\sigma_0(x) = x$) and σ_n, ($n > 1$) positive maps satisfying the relation

$$\sigma_1(\sigma_n(x)) = p_1 \sigma_{n+1}(x) + p_2 \sigma_{n-1}(x) \qquad n \geqslant 1, \quad x \in A$$

for some real numbers p_1, p_2, $0 < p_1 \leqslant 1$ and $p_1 + p_2 = 1$. The elements σ_n commute with one another, i.e. $\sigma_n \circ \sigma_k = \sigma_k \circ \sigma_n$, since σ_n is just a polynomial in σ_1 and σ_0 the identity map. Also, if σ_n, ($n \geqslant 2$) are positive they are automatically completely positive and normal since they are linear combinations of σ_0 and powers of σ_1. For our theorems, we also require the existence of a faithful normal state ρ, invariant under the action of σ_1.

Our results then focus around the question of when and how the sums

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} \sigma_k$$

converge. It is shown in this paper that if the spectrum of the bounded operator given by the action of σ_1 on the left Hilbert algebra $L^2(A, \rho)$ is contained in a certain set D_{p_1} (note the dependence on p_1), and this operator is normal, then $S_n(x)$ converges almost uniformly to an element $\hat{x} \in A$. In the case $p_1 = 1$ and σ_1 a *-automorphism, this is Lance's non-commutative ergodic theorem. Kummerer [Ku] proved a theorem similar to the above for $p_1 = 1$ and σ_1 a positive normal map. In that case, one can remove the restriction that the action on the left Hilbert algebra is normal.

1.3. *A Note on the Method of Presentation*

The theorems of immediate interest are those concerning the free group actions on the von Neumann algebra which are merely corollaries to the more general case. However, to indicate their major status to the paper, they are stated as theorems before the general case theorem. This is not to diminish the importance of the general case, as its "random walk" appearance seems quite interesting on its own.

2. ERGODIC THEOREMS

2.1. *A von Neumann Type Ergodic Theorem*

2.1.1. *Technical Lemmas*

The following two lemmas are needed for the final theorems in this section. We introduce two sets indexed by a real number p, $0<p\leqslant 1$. These sets, denoted E_p and D_p, are sets in the complex plane. They are defined as follows.

$$E_p=\{z\in\mathbf{C}\mid |\sqrt{z+4p-4p^2}+\sqrt{z-4p+4p^2}|<2\sqrt{p} \text{ and } |\sqrt{z+4p-4p^2}-\sqrt{z-4p+4p^2}|<2\sqrt{p}\} \tag{2}$$

$$D_p=\{z\in\mathbf{C}\mid |\sqrt{z+4p-4p^2}+\sqrt{z-4p+4p^2}|\leqslant 2\sqrt{p} \text{ and } |\sqrt{z+4p-4p^2}-\sqrt{z-4p+4p^2}|\leqslant 2\sqrt{p}\} \tag{3}$$

To give the reader some intution as to the shape of these sets, we provide in Fig. 1 four graphs of the set D_p for $p=1/2$, $2/3$, $3/4$, and 1. Note that when $p<\frac{1}{2}$, $D_p=\{1\}\cup\{-1\}$ and E_p is empty. If $\frac{1}{2}<p\leqslant 1$ then D_p is the closure of E_p. If $p=1/2$, then $D_p=[-1, 1]$ and E_p is empty.

LEMMA 1. *Given $0<p_1\leqslant 1$, $p_1+p_2=1$, f_n functions on the complex plane such that $f_0(z)=1$, $f_1(z)=z$, and $zf_n(z)=p_1 f_{n+1}(z)+p_2 f_{n-1}(z)$ for $n\geqslant 1$, then*

$$\lim_{n\to\infty} f_n(z)$$

exists if and only if $z\in E_{p_1}\cup\{1\}$. The limit is zero on E_{p_1} and one on $\{1\}$. Moreover, there exists an N such that if $n>N$ then $|f_n(z)|\leqslant 1$ on E_{p_1}.

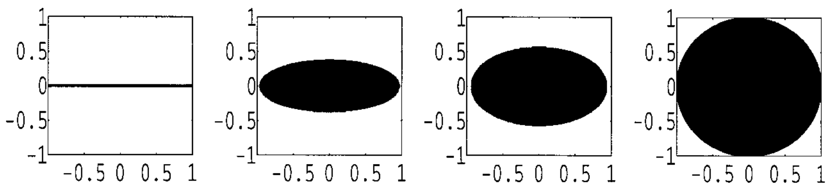

FIG. 1. $D_{1/2}, D_{2/3}, D_{3/4}, D_1$.

LEMMA 2. *With f_n defined as above, then $n^{-1}\sum_{k=0}^{n-1} f_k(z)$ converges pointwise if and only if $z \in D_{p_1}$. It converges to zero for $z \neq 1$ and to one if $z=1$. Moreover, there exists an N such that if $n>N$ then $|n^{-1}\sum_{k=0}^{n-1} f_k(z)| \leqslant 1$ on D_{p_1}.*

The proofs of the previous lemmas are combinatorial in nature with many special cases. Inclusion of the proofs here would distort the flow of the paper and hence they are relegated to Appendix A.

2.1.2. *A von Neumann Type Theorem*

The free group case. The following ergodic theorems resemble the ergodic theorems of Guivarc'h and von Neumann.

THEOREM 3. *Given $\{u_i\}_{i=1}^r$ unitaries acting on a Hilbert space H, F_r the free group on r generators $\{a_i\}_{i=1}^r$, w_n the set of words of length n in F_r, and $|w_n|$ the number of elements of w_n. Let ϕ be the group homomorphism from F_r to the unitaries of $B(H)$ given by $\phi(a_i)=u_i$, $H_1=\{\eta \mid t_1\eta=\eta\}$, $H_{-1}=\{\eta \mid t_1\eta=-\eta\}$, and define self adjoint contractions t_n by $t_n=|w_n|^{-1}\sum_{a\in w_n}\phi(a)$. The sequence $t_n|_{H_{-1}^{\perp}}$ converges strongly to the projection onto H_1.*

We also get the following theorem for the sequence $n^{-1}\sum_{k=0}^{n-1} t_k$.

THEOREM 4. *Given t_k and H_1 as above, then $n^{-1}\sum_{k=0}^{n-1} t_k$ converges strongly to the projection onto H_1.*

We prove these theorems by stating and proving a more general case below.

The general case. Given H a Hilbert space, x_1 a normal operator in $B(H)$, $x_0=I$, p_1, p_2 real numbers, $0<p_1\leqslant 1$, $p_1+p_2=1$, then define normal operators x_n $(n>1)$ by use of the relation

$$x_1 x_n = p_1 x_{n+1} + p_2 x_{n-1} \qquad (n \geqslant 1).$$

This suggests a random walk with a probability p_1 of moving forward and p_2 of moving backwards. We give convergence results for the sequence x_n and the ergodic sum $n^{-1}\sum_{k=0}^{n-1} x_k$.

We use standard notation as in [KR]. Given x a normal operator in $B(H)$, let $\sigma(x)$ denote the spectrum of x. The functional calculus gives a decomposition

$$x = \int_{\sigma(x)} \lambda \, dE(\lambda). \tag{4}$$

The spectral projection of an operator x for a borel set $S \subset \sigma(x)$ is denoted by $E(x; S)$.

THEOREM 5. *Given a Hilbert space H and a normal operator x_1 in $B(H)$, $x_0 = I$, p_1, p_2 real numbers such that $p_1 + p_2 = 1$, $0 < p_1 \leqslant 1$, then define x_n $(n > 1)$ via the relation $x_1 x_n = p_1 x_{n+1} + p_2 x_{n-1}$. If $H_1 = \{\eta \mid x_1\eta = \eta\}$ and $H_{-1} = \{\eta \mid x_1\eta = -\eta\}$ then $x_n|_{H_{-1}^{\perp}}$ converges strongly if the spectral projection $E(x_1; S) = 0$ where $S = \sigma(x_1) \backslash (E_{p_1} \cup \{1\})$. It converges to P_{H_1}, the projection onto H_1.*

Proof. Let $dE(\lambda)$ denote the spectral measure of x_1 so that

$$x_1 = \int_{\sigma(x_1)} \lambda \, dE(\lambda).$$

As we can express x_n in terms of I and x_1, then there are functions f_n such that

$$x_n = \int_{\sigma(x_1)} f_n(\lambda) \, dE(\lambda).$$

Since $z f_n(z) = p_1 f_{n+1}(z) + p_2 f_{n-1}(z)$ we will be able to use results from Lemma 1.

Note that H_1 and H_{-1} are invariant under both x_1 and x_1^* since if $\eta \in H_{-1}$ then $x_1(x_1^*\eta) = x_1^* x_1 \eta = -x_1^*\eta$ and likewise for $\eta \in H_1$. This makes H_1, H_{-1}, $H_1^{\perp}$ and $H_{-1}^{\perp}$ invariant under x_n.

Let $\chi_{\sigma(x_1)\backslash\{-1\}}$ be the characteristic function of $\sigma(x_1)\backslash\{-1\}$. As $H_{-1}^{\perp}$ is an invariant subspace under x_n then the restriction of the operator to $H_{-1}^{\perp}$ gives

$$x_n|_{H_{-1}^{\perp}} = \int_{\sigma(x_1)} f_n(\lambda) \, \chi_{\sigma(x_1)\backslash\{-1\}} \, (\lambda) \, dE(\lambda)|_{H_{-1}^{\perp}}.$$

If P_{H_1} is the projection onto H_1, then as H_1 is invariant under x_n and $f_n(1) = 1$ we have

$$(x_n - P_{H_1})|_{H_{-1}^{\perp}} = \int_{\sigma(x_1)} f_n(\lambda) \, \chi_{\sigma(x_1)\backslash\{1,\,-1\}}(\lambda) \, dE(\lambda)|_{H_{-1}^{\perp}}.$$

From Lemma 1 we know that $f_n(z)$ converges pointwise if and only if $z \in E_{p_1} \cup \{1\}$ and that for some $M > 0$, $f_n(z) \leqslant M$ for all n. It converges to zero on E_{p_1} and hence we get that $(x_n - P_{H_1})|_{H_{-1}^{\perp}}$ goes to zero in the strong operator topology if the spectral projection corresponding to S is zero. ∎

THEOREM 6. *Given x_n as in the theorem above, then the partial sums $n^{-1}\sum_{k=0}^{n-1} x_k$ converge strongly if and only if the spectral projection $E(x_1; \hat{S})=0$, where $\hat{S}=\sigma(x_1)\backslash D_{p_1}$. It converges to P_{H_1}, the projection onto the set $H_1=\{\eta\in H \mid x_1\eta=\eta\}$.*

Proof. The proof is similar to the above, except that Lemma 2 is used instead of Lemma 1. ∎

3. A VON NEUMANN TYPE ERGODIC THEOREM FOR VON NEUMANN ALGEBRAS

3.1. *Notation and Definitions*

We are now able to turn our attention to completely positive normal maps acting on a von Neumann algebra. Our final result in Section 5 is much stronger than the following result; however, we need it in order to prove our final theorem.

Given A a von Neumann algebra, ρ a faithful normal state, we get a Hilbert space $H=L^2(A,\rho)$, a separating cyclic vector ζ and a mapping $\pi: A\to B(H)$. Let $H^o=\pi(A)\zeta$, which is a dense subset of H.

We note that any normal completely positive map σ_1 that leaves ρ invariant induces a linear operator $\Pi(\sigma_1)$ on H^o by $\Pi(\sigma_1)\,\pi(x)\zeta=\pi(\sigma_1(x))\zeta$. If $\sigma_1(1)=1$, then since σ_1 is completely positive we have that $\Pi(\sigma_1)$ is bounded. This is from a well known application of Stinespring's theorem, which we sketch for completeness. If v is the isometry which comes from the Stinespring theorems representation of σ_1, then $\sigma_1(x)^*\,\sigma_1(x)=v^*x^*vv^*xv\leqslant v^*x^*xv=\sigma_1(x^*x)$. This gives that, for $x\in A$, $\|\Pi(\sigma_1)\,x\zeta\|^2\leqslant\rho(\sigma_1(x^*x))=\rho(x^*x)=\|x\zeta\|^2$. As H^o is dense, and $\Pi(\sigma_1)$ is contractive on H^o, then $\Pi(\sigma_1)$ extends to a contractive map everywhere.

A large part of this paper deals with sequences of unital completely positive maps $\sigma_n: A\to A$ and sums of these maps, S_n, satisfying

$$0<p_1\leqslant 1, \qquad p_1+p_2=1 \tag{5}$$

$$\sigma_0(x)=x \tag{6}$$

$$\sigma_1(\sigma_n(x))=p_1\sigma_{n+1}(x)+p_2\sigma_{n-1}(x) \qquad (n\geqslant 1) \tag{7}$$

$$S_n(x)=n^{-1}\sum_{k=0}^{n-1}\sigma_k(x). \tag{8}$$

It is important to note that in the case of free group actions on the von Neumann algebra, for the completely positive maps σ_n defined by (1) in Section 1.2 we have that $\Pi(\sigma_n)$ are self-adjoint contractions since

*-automorphisms which leave ρ invariant map to unitaries acting on $L^2(A, \rho)$, and the set w_n is closed with respect to inverses. Also, the free group actions give σ_n satisfying our above relations with $p_1 > 0.5$ and hence $\sigma(\Pi(\sigma_1)) \subset D_{p_1}$ (as $[-1, 1]$ is contained in D_{p_1}). Hence, all of the following results apply to the free group actions on the von Neumann algebra, as long as there is a faithful normal state invariant under the action of the *-automorphisms which comprise the free group actions.

3.2. *A von Neumann Type Ergodic Theorem*

THEOREM 7. *Let $\{\alpha_i\}_{i=1}^{r}$ be *-automorphism of a von Neumann algebra A which leave a faithful normal state ρ invariant. Let F_r be the free group on r generators $\{a_i\}_{i=1}^{r}$, and let $\phi: F_r \to \mathrm{Aut}(A)$ be the group homomorphism defined by $\phi: a_i \mapsto \alpha_i$. Let w_n be the set of all words of F_r of length n, $|w_n|$ the number of elements of w_n, $\sigma_n = |w_n|^{-1} \sum_{a \in w_n} \phi(a)$, and $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$. If $x \in A$, then $S_n(x)$ converges ultrastrongly to an element $\hat{x}$ in A.*

This theorem is a corollary of the theorem below.

THEOREM 8. *Let σ_1 be a normal completely positive map of a von Neumann algebra A, $\sigma_1(I) = I$, ρ a σ_1-invariant faithful normal state, $p_1 \geqslant p_2 \geqslant 0$, $p_1 + p_2 = 1$, and σ_n $(n \geqslant 2)$ positive maps such that $\sigma_1 \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$, where $\sigma_0(x) = x$. Let $\Pi(\sigma_1)$ be the action of σ_1 on $L^2(A, \rho)$ given by $\Pi(\sigma_1)\, \pi(x)\zeta = \pi(\sigma_1(x))\zeta$, where ζ is the cyclic separating vector from the GNS with respect to ρ. If $\Pi(\sigma_1)$ is a normal operator such that the spectrum $\sigma(\Pi(\sigma_1))$ is contained in D_{p_1}, defined in (3), then for $x \in A$, $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$ converges ultrastrongly to an element $\hat{x} \in A$.*

Proof. As $\sigma_1(1) = 1$, then $\sigma_k(1) = 1$ which gives $\|\sigma_k(x)\| \leqslant \|x\|$ as σ_k are completely positive and thus $\|S_n(x)\| \leqslant \|x\|$. Hence it suffices to show strong convergence. From our discussion of Stinespring's theorem in Section 3.1, $\Pi(\sigma_1)$ is bounded and hence $\|\Pi(S_n)\| \leqslant 1$ for all n.

As $\Pi(S_n)$ are normal operators on H then by Theorem 6 we have the existence of a projection P such that

$$\pi(S_n(x))\zeta = \Pi(S_n)\pi(x)\zeta \to P\pi(x)\zeta.$$

For any $y' \in \pi(A)'$ we have that $\pi(S_n(x))\, y'\zeta$ converges. As ζ is separating, $\pi(A)'\zeta$ is dense and as $\pi(S_n(x))$ is uniformly bounded by $\|x\|$ by our discussion of Steinspring's theorem in section 3.1, we have that $\pi(S_n(x))$ converges strongly. As the representation is with respect to a faithful normal state π, we note that $S_n(x)$ converges strongly to an element we call $\hat{x}$. ∎

We also have the following statement which parallels a theorem found in [L].

THEOREM 9. *Under all the hypothesis of the above theorem, define* $S_n(\psi)$ *as* $S_n(\psi)(x)=\psi(S_n(x))=\psi(n^{-1}\sum_{k=0}^{n-1}\sigma_k(x))$. *For* $\psi\in A_*$, *the predual of* A, $S_n(\psi)$ *converges in norm to* $\hat{\phi}\in A_*$, *where* $\hat{\phi}(x)=\phi(\hat{x})$.

Proof. Define a linear functional $\omega_{\eta,\gamma}$ by $\omega_{\eta,\gamma}(x)=\langle x\eta,\gamma\rangle$.
It suffices to show the result for linear functionals of the form $\omega_{x'\zeta,y'\zeta}$ for $x',y'\in A'$, since ζ is separating for A.
Now,

$$\begin{aligned}
&\|S_n(\omega_{x'\zeta,y'\zeta})-\hat{\omega}_{x'\zeta,y'\zeta}\|\\
&\quad=\sup_{z\in A,\ \|z\|=1}|\langle(\pi(S_n(z))-\pi(\hat{z}))\,x'\zeta,y'\zeta\rangle|\\
&\quad=\sup_{z\in A,\ \|z\|=1}|\langle(\Pi(S_n)-P)\,\pi(z)\zeta,(x')^*\,y'\zeta\rangle|\\
&\quad=\sup_{z\in A,\ \|z\|=1}|\langle z\zeta,(\Pi(S_n)^*-P)(x')^*\,y'\zeta\rangle|\\
&\quad\leqslant\|(\Pi(S_n)^*-P)(x')^*\,y'\zeta\|\to 0
\end{aligned}$$

as $\Pi(S_n)^*$ is a normal operator created from $\Pi(\sigma_n)^*$, $\sigma(\Pi(\sigma_1)^*)\subset D_{p_1}$, and $\sigma(\Pi(\sigma_n))^*$ satisfies the relations $\Pi(\sigma_1)^*\,\Pi(\sigma_n)^*=p_1\Pi(\sigma_{n+1})^*+p_2\Pi(\sigma_{n-1})^*$, the $\Pi(S_n)^*\to P$ in the strong operator topology by Theorem 8. ▮

LEMMA 10. *With the hypothesis of Theorem* 9, *if* $x_n\to x$ *strong-*, then* $\hat{x}_k$, *the strong limit of* $S_n(x_k)$ (*over* n) *converges strong-* to* $\hat{x}$, *the strong limit of* $S_n(x)$.

Proof. As in the proof of Theorem 8, there exists a projection P such that $\|\pi(\hat{x}_k-\hat{x})\zeta\|=\|P\pi(x_k-x)\zeta\|$ and hence $\|\pi(\hat{x}_k-\hat{x})\zeta\|\to 0$ as $\|\pi(x_k-x)\zeta\|\to 0$. Since ζ is separating, we have that $\pi(\hat{x}_k-\hat{x})\to 0$ strongly, and as π is normal, then $\hat{x}_k-\hat{x}\to 0$ strongly. Finally, by repeating the above argument with adjoints gives the strong-* convergence. ▮

4. A DENSITY RESULT

The results in this section are necessary for the final theorem of this paper. They also have interest for their own sake and spawn several interesting questions, in addition to providing a level of intuition about the convergence of these ergodic sums. We first need two technical lemmas.

4.1. *Technical Lemmas*

LEMMA 11. *If* $f_n(z) = 1 - ((z+1)/2)^n$, $\hat{f}_n(j)$ *the* j*th polynomial coefficient* ($f_n(z) = \hat{f}_n(0) + \hat{f}_n(1)z + \cdots + \hat{f}_n(n)z^n$), *then*

(i) $f_n(1) = 0$ *and if* χ_1 *is the characteristic function at* 1, *then* $f_n + \chi_1$ *approaches* 1 *pointwise for* $z \in \mathbf{C}$, $|z| \leqslant 1$.

(ii) $\hat{f}_n = \{\hat{f}_n(i)\}_{i=0}^{\infty}$ *is a sequence in* $l_1(Z^+)$ *such that* $\|\hat{f}_n\|_1 \leqslant 2$ *for all* n.

Proof. (i) If $|z| \leqslant 1$ and $z \neq 1$ then

$$\left|\frac{z+1}{2}\right| < 1.$$

(ii) Let $g_n(z) = ((z+1)/2)^n$, and $\hat{g}_n$ be the polynomial coefficients of g_n (note they are all positive).

$$\begin{aligned}
\|\hat{f}_n\|_1 &= |\hat{f}_n(0)| + |\hat{f}_n(1)| + \cdots + |\hat{f}_n(n)| \\
&\leqslant 1 + |\hat{g}_n(0)| + |\hat{g}_n(1)| + \cdots + |\hat{g}_n(n)| \\
&= 1 + \hat{g}_n(0) + \hat{g}_n(1) + \cdots + \hat{g}_n(n) \\
&= 1 + \hat{g}_n(0) + \hat{g}_n(1)1 + \cdots + \hat{g}_n(n)\, 1^n \\
&= 1 + g_n(1) = 2. \quad \blacksquare
\end{aligned}$$

LEMMA 12. *Let* σ_1 *be a normal completely positive map of a von Neumann algebra* A, $\sigma_1(1) = 1$, $\sigma_0(x) = x$, p_1, p_2 *real numbers such that* $0 < p_1 \leqslant 1$ *and* $p_1 + p_2 = 1$, *and* σ_n ($n \geqslant 2$) *positive maps which satisfy the relation* $\sigma_1 \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$ *for* $n \geqslant 1$. *If for* $k > 0$ *we define* $\sigma_{-k} = \sigma_k$, *then the following statements are true.*

(i) $\sigma_1 \sigma_0 = \sigma_{-1}$

(ii) *For* $k > 0$, $\sigma_1 \sigma_k = p_1 \sigma_{k+1} + p_2 \sigma_{k-1}$

(iii) *For* $k < 0$, $\sigma_1 \sigma_k = p_2 \sigma_{k+1} + p_1 \sigma_{k-1}$

Moreover, if we use the multiplication rules above to define unique coefficients $a_{i,r,k} \geqslant 0$, $0 \leqslant i \leqslant r$ *such that*

$$\sigma_1^r \sigma_k = a_{0,r,k} \sigma_{k+r} + a_{1,r,k} \sigma_{k+r-2} + \cdots + a_{r,r,k} \sigma_{k-r}$$

then $\sum_{i=0}^r a_{i,r,k} = 1$. *Finally, we note that if* k_1, $k_2 > r+1$ *then* $a_{i,r,k_1} = a_{i,r,k_2}$.

Proof. Obvious from the definitions. $\blacksquare$

4.2. *The Density Result*

THEOREM 13. *Let $\{\alpha_i\}_{i=1}^r$ be *-automorphisms of a von Neumann algebra A which leave a faithful normal state ρ invariant. Let F_r be the free group on r generators $\{a_i\}_{i=1}^r$, and let $\phi: F_r \to \mathrm{Aut}(A)$ be the group homomorphism defined by $\phi: a_i \mapsto \alpha_i$. Let w_n be the set of all words of F_r of length n, $|w_n|$ the number of elements of w_n, $\sigma_n = |w_n|^{-1} \sum_{a \in w_n} \phi(a)$, and $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$. The convex set $A_u = \{x \in A \mid \|S_n(x) - \hat{x}\|. \to 0\}$ is strong-* dense in A, where $\hat{x}$ is the strong limit of $S_n(x)$ shown to exist in Section 3.*

The above theorem is once again a corollary to a more general result.

THEOREM 14. *Let σ_1 be a normal completely positive map of a von Neumann algebra A, $\sigma_1(I) = I$, ρ a σ_1-invariant faithful normal state, $p_1 \geqslant p_2 \geqslant 0$, $p_1 + p_2 = 1$, and σ_n $(n \geqslant 2)$ positive maps such that $\sigma_1 \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$, where $\sigma_0(x) = x$. Let $\Pi(\sigma_1)$ be the action of σ_1 on $L^2(A, \rho)$ given by $\Pi(\sigma_1)\, x\zeta = \pi(\sigma_1(x))\zeta$, where ζ is the cyclic separating vector from the GNS w.r.t ρ, and define $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$. If $\Pi(\sigma_1)$ is a normal operator such that the spectrum $\sigma(\Pi(\sigma_1))$ is contained in D_{p_1}, defined in (3), then the convex set $A_u = \{x \in A \mid \|S_n(x) - \hat{x}\| \to 0\}$ is strong-* dense in A, where $\hat{x}$ is the strong limit of $S_n(x)$ shown to exist in Section 3.*

Proof. Convexity of A_u is trivial.

Given $x \in A$ we wish to (i) find a sequence of elements in A that converge to x strong-* and (ii) show that this sequence is in A_u. Again we use standard terminology. π is the representation of A into $B(H)$, Λ is the map from A into H given by $x \mapsto \pi(x)\zeta$, where ζ is the cyclic, separating vector which arises from the GNS representation with respect to ρ.

(i) As $\Pi(\sigma_1)$ is a normal operator on H, we have the following decomposition

$$\Pi(\sigma_1) = \int_{\sigma(\Pi(\sigma_1))} \lambda \, dE(\lambda).$$

From Theorem 4, if χ_1 the characteristic function at $\{1\}$, $P =$ projection onto $\{\eta \mid \Pi(\sigma_1)\, \eta = \eta\} = \int_{\sigma(\Pi(\sigma_1))} \chi_1 \, dE(\lambda)$, then if $\gamma \in H$, $\|(\Pi(S_n) - P)\gamma\| \to 0$.

Let $f_n(z) = 1 - ((z+1)/2)^n$, $\hat{f}_n(j)$ the jth polynomial coefficient ($f_n(z) = \hat{f}_n(0) + \hat{f}_n(1) z + \cdots + \hat{f}_n(n)\, z^n$). Define $x_n = \hat{f}_n(0) + \hat{f}_n(1)\, \sigma_1(x) + \cdots + \hat{f}_n(n)\, \sigma_1^n(x)$. Any f_n satisfying conditions (i) and (ii) of Lemma 11 will suffice, which we note because this freedom of f_n's might be helpful in exploring properties of A_u.

If ζ is the cyclic vector obtained from the identity in the GNS, (hence $\pi(x)\zeta = \Lambda(x)$), then by (i) of Lemma 11,

$$\pi(x_n+\hat{x})\zeta = \left(\int_{\sigma(\Pi(\sigma_1))} f_n(\lambda)+\chi_1(\lambda)\, dE(\lambda)\right)\pi(x)\zeta \tag{9}$$

$$\to \pi(x)\zeta. \tag{10}$$

Note that as ζ is a cyclic vector for the algebra $\pi(A)'$, and $x_n+\hat{x}$ are uniformly bounded by Lemma 4 (ii), then the convergence of $\pi(x_n+\hat{x})$ to $\pi(x)$ is strong. As the representation is with respect to a faithful normal state, then $x_n+\hat{x}$ converges strongly to x. As it converges under involution of both sides, it converges strong-* to x.

(ii) We show that each of these elements is in A_u by showing that

$$S_n(x_l+\hat{x})$$

converges in norm (as $n\to\infty$) to $\hat{x}$ for all l. As $S_n(\hat{x})$ is $\hat{x}$, it is left to show that the term $S_n(x_l)$ goes to zero in norm as n approaches infinity.

We adopt the notation of Lemma 12. For $k>0$, we define $\sigma_{-k}=\sigma_k$ and we have coefficients $a_{i,r,k}\geqslant 0$, $0\leqslant i\leqslant r$ such that

$$\sigma_1^r\sigma_k = a_{0,r,k}\sigma_{k+r}+a_{1,r,k}\sigma_{k+r-2}+\cdots+a_{r,r,k}\sigma_{k-r}$$

and $\sum_{i=0}^r a_{i,r,k}=1$. Also, if $k_1, k_2>r+1$ then $a_{i,r,k_1}=a_{i,r,k_2}$.

We now show the norm convergence of $S_n(x_l)$. We remind the reader that $\sigma_k\circ\sigma_n=\sigma_n\circ\sigma_k$.

$$\begin{aligned} S_n(x_l) &= \frac{1}{n}\sum_{k=0}^{n-1}\sigma_k\left(\sum_{r\in Z^+}\hat{f}_l(r)\,\sigma_1^r\right)(x) \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{r\in Z^+}\hat{f}_l(r)\,\sigma_1^r\sigma_k(x) \\ &= \frac{1}{n}\sum_{r\in Z^+}\hat{f}_l(r)\sum_{p=0}^{r}\sum_{k=0}^{n-1}a_{p,r,k}\sigma_{|k+r-2p|}(x). \end{aligned} \tag{11}$$

As $\{\hat{f}_l(i)\}$ is l^1 (from (ii) of our previous lemma) and $\sum_{i=0}^\infty \hat{f}_l(i)=0$ (since $f_l(1)=0$), then given $\varepsilon>0$ one can choose m such that $\sum_{r=m+1}^\infty |\hat{f}_l(r)|<\varepsilon$, and hence $|\sum_{r=0}^m \hat{f}_l(r)|<\varepsilon$. We note that our specific function f_l, this is obvious as it is a polynomial, however as we are only requiring the strength of (i) and (ii) of Lemma 11, we prove for the more general, possibly non-polynomial f_n. We now break up the sum in (11) into the two sums

$$\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}\sum_{k=0}^{n-1}a_{p,r,k}\sigma_{|k+r-2p|}(x) \tag{12}$$

and

$$\frac{1}{n}\sum_{r=m}^{\infty}\hat{f}_l(r)\sum_{p=0}^{r}\sum_{k=0}^{n-1}a_{p,r,k}\sigma_{|k+r-2p|}(x). \tag{13}$$

It will be shown that the limit of the norm of each sum is less than a constant times $\varepsilon\,\|x\|$, but first the sum in (12) is dealt with. We assume for the rest of this proof that $n\geqslant 2m+2$. This is admissible as we are taking the limit of n. We decompose (12) into the sums

$$\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}\sum_{k=r+1}^{n-1}a_{p,r,k}\sigma_{k+r-2p}(x) \tag{14}$$

and

$$\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}\sum_{k=0}^{r}a_{p,r,k}\sigma_{|k+r-2p|}(x) \tag{15}$$

It is clear that the norm of the sum in (15) can be made less than $\varepsilon\,\|x\|$ as it is a finite sum divided by n as n goes to infinity. We need only focus on (14). We appeal to the fact that for $k_1, k_2>r+1$, then $a_{p,r,k_1}=a_{p,r,k_2}$ and so we let $a_{p,r}$ denote $a_{p,r,k}$ for $k>r$ which allows the passing of the coefficients $a_{p,r,k}$ through the sum over k. (14) becomes

$$\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}\sum_{k=r+1}^{n-1}\sigma_{k+r-2p}(x).$$

For fixed $p\leqslant r$, fixed $r\leqslant m$, then

$$\sum_{k=r+1}^{n-1}\sigma_{k+r-2p}=\sigma_{2r+1}+\sigma_{2r+2}+\cdots+\sigma_{n-r-1}+Q_{r,p}, \tag{16}$$

where $Q_{r,p}$ has $2r$ terms, and so $\|Q_{r,p}(x)\|\leqslant 2r\,\|x\|$ since $\sigma_k(x)\leqslant\|x\|$. We then get that

$$\begin{aligned}\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}\sum_{k=r+1}^{n-1}\sigma_{k+r-2p}(x)\right\| &\leqslant\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})(x)\right\| \\ &\quad+\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}Q_{r,p}(x)\right\| \\ &\leqslant\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})(x)\right\| \\ &\quad+\frac{2m^3}{n}\,\|x\|\,\|\hat{f}_l\|_1.\end{aligned} \tag{17}$$

It is clear that $(2m^3/n)\|x\|\,\|\hat{f}_l\|_1$ goes to zero as n becomes large, hence we need to concentrate on the term

$$\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}(\sigma_{2r+1}+\sigma_{2r+2}+\cdots+\sigma_{n-r-1})(x)\right\|$$

and show that this, in limit, tends towards zero. By Lemma 12 we have that the sum of the coefficients is one and hence

$$\begin{aligned}\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\sum_{p=0}^{r}a_{p,r}(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})(x)\right\|\\ =\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})(x)\right\|\end{aligned} \tag{18}$$

Again, collecting a common term from (18) will give convergence to zero. For $0\leqslant r\leqslant m$,

$$(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})=\sigma_{2m+1}+\cdots+\sigma_{n-m-1}+V_r, \tag{19}$$

where V_r has $3m-3r$ terms and so $\|V_r(x)\|<3m\,\|x\|$. This allows us to decompose (18).

$$\begin{aligned}&\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)(\sigma_{2r+1}+\cdots+\sigma_{n-r-1})(x)\right\|\\ &\leqslant\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)(\sigma_{2m+1}+\cdots+\sigma_{n-m-1})(x)\right\|+\left\|\frac{1}{n}\sum_{r=0}^{m}\hat{f}_l(r)\,V_r(x)\right\|\\ &\leqslant\frac{1}{n}\left(\left|\sum_{r=0}^{m}\hat{f}_l(r)\right|\right)\|(\sigma_{2m+1}+\cdots+\sigma_{n-m-1})(x)\|+\frac{3m^2}{n}\|x\|\,\|\hat{f}_l\|_1\\ &\leqslant\varepsilon\left(\frac{n-3m-2}{n}\right)\|x\|+\frac{3m^2}{n}\|x\|\leqslant\left(\varepsilon+\frac{3m^2}{n}\|\hat{f}_l\|_1\right)\|x\|.\end{aligned} \tag{20}$$

We now have that (12) is less than $(2\varepsilon+(3m^2/n)\|\hat{f}_l\|_1+(2m^3/n)\|\hat{f}_l\|_1)\|x\|$ (combine (15), (17), and (20)), which can be made less than $3\varepsilon\,\|x\|\,\|\hat{f}_l\|_1$ by making n large.

The final piece that needs to be shown is that (13) goes to zero in norm. We proceed immediately to the proof.

$$\begin{aligned}
&\left\| \frac{1}{n} \sum_{r=m+1}^{\infty} \hat{f}_l(r) \sum_{p=0}^{r} \sum_{k=0}^{n-1} a_{p,r,k} \sigma_{|k+r-2p|}(x) \right\| \\
&\quad \leqslant \frac{1}{n} \sum_{r=m+1}^{\infty} |\hat{f}_l(r)| \sum_{p=0}^{r} \sum_{k=0}^{n-1} |a_{p,r,k}| \, \|\sigma_{|k+r-2p|}(x)\| \\
&\quad \leqslant \sum_{r=m+1}^{\infty} |\hat{f}_l(r)| \sum_{p=0}^{r} a_{p,r,k} \, \|x\| \\
&\quad = \sum_{r=m+1}^{\infty} |\hat{f}_l(r)| \, \|x\| \leqslant \varepsilon \, \|x\| \qquad (21)
\end{aligned}$$

Combining (17), (20), and (21) we get that, for sufficiently large n,

$$\left\| \frac{1}{n} \sum_{r \in Z^+} \hat{f}_l(r) \sum_{p=0}^{r} a_{p,r} \sum_{k=0}^{n-1} \sigma_{|k+r-2p|}(x) \right\| \leqslant 4\varepsilon M \, \|x\|,$$

where M is the supremum of 1 or $\|\hat{f}_l\|_1$, which exists by (ii) of our previous lemma. ∎

An interesting statement, which is trivial in the case studied in [L], is the following. (Note the weaker hypothesis than in previous theorems).

Remark 1. Let σ_1 be a map of a von Neumann algebra A, $\sigma_1(1) = 1$, ρ a σ_1-invariant faithful normal state, $p_1 > p_2 \geqslant 0$, $p_1 + p_2 = 1$, and σ_n $(n \geqslant 2)$ positive maps such that $\sigma_1 \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$, where $\sigma_0(x) = x$. For x in the kernel of σ_1, $S_n(x)$ converges to zero in norm.

The proof of the above remark is immediate from the relation $\sigma_k = (1/p_1)(\sigma_{k-1}\sigma_1 - p_2 \sigma_{k-2})$, which gives $\sigma_k(x) = -(p_2/p_1)\, \sigma_{k-2}(x)$. Hence, we have the following reduction.

$$\sigma_k(x) = \begin{cases} \left(-\dfrac{p_2}{p_1}\right)^{k/2} x & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

As $p_2 \neq p_1$, $\|S_{2n}(x)\| = \|(1/2n) \sum_{k=0}^{n-1} (-p_2/p_1)^k x\| \leqslant (1/2n)(p_1/(p_1 - p_2))$ $\|x\| \to 0$.

There is also an open question first posed by Lance for the automorphism case: does there exist a weakly dense C* subalgebra of A_u? If not, are there normal completely positive maps σ_n such that they satisfy our assumptions and there are no non-trivial C* subalgebras of A_u?

5. ALMOST UNIFORM CONVERGENCE OF ERGODIC SUMS

5.1. *A Maximal Ergodic Theorem*

The following is a crucial bounding lemma which relates sums of our free group actions to traditional ergodic sums.

LEMMA 15. *Let σ_k be defined as in Theorem* 14. *There exists a constant C_{p_1} such that if $x \in A$ is a positive operator, then*

$$\frac{1}{n}\sum_{k=0}^{n-1} \sigma_k(x) \leqslant C_{p_1} \frac{1}{3n} \sum_{k=0}^{3n-1} \sigma_1^k(x).$$

Proof. This is the non-commutative extension of a lemma of A. Nevo and E. Stein (Lemma 1 in [NS]). The proof of the commutative case carries over to the present situation. ▌

For completeness, we state a theorem of Goldstein's which generalized the maximal ergodic theorem of Lance. For a full discussion and the proof, we reference to [J].

THEOREM (2.2.15 of J). *Let A be a von Neumann algebra with a faithful normal state ϕ, and let α be a normal positive map of A such that $\alpha(I) \leqslant I$ and $\phi(\alpha(x)) \leqslant \phi(x)$, for all $x \in A$, $x > 0$. Then, if $I \geqslant x \geqslant 0$, $x \in A$, there exists a $c \in A$, $c > 0$ such that $\|c\| \leqslant 4$, $\phi(c) \leqslant 8\phi(x)^{1/2}$, and $s_n(x) = n^{-1} \sum_{k=0}^{n-1} \alpha^k(x) \leqslant c$ for $n = 1, 2, \ldots$.*

We now use this theorem along with our preceeding lemma to get our own maximal ergodic theorem.

LEMMA 16. *Under all the hypothesis of Theorem* 14, *then there is a constant C_{p_1} such that for $x \in A$, $1 \geqslant x \geqslant 0$, there exists a $y \in A$, such that $0 < y < 4C_{p_1} I$, $\rho(y) \leqslant 8(C_{p_1}\rho(x))^{1/2}$ and $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x) \leqslant y$ for all $n = 1, 2, \ldots$.*

Proof. Take C_{p_1} as in Lemma 15 above. The maximal ergodic theorem in [J] (stated above), gives that there is a $c > 0$ such that $\|c\| \leqslant 4$, $\phi(c) \leqslant 8\phi(x)^{1/2}$ and

$$\frac{1}{3n} \sum_{k=0}^{3n-1} \sigma_1^k(x) \leqslant c$$

for all n. Using Lemma 15 gives that

$$S_n(x) \leqslant \frac{C_{p_1}}{3n} \sum_{k=0}^{3n-1} \sigma_1^k(x) \leqslant C_{p_1} c.$$

Taking $y = C_{p_1} c$ completes the proof. ∎

5.2. *The Almost Uniform Theorem*

We have now generalized the major theorems needed to prove our final result. The rest of the theorems are now relatively easy alterations of Theorems 5.6 and 5.7 in [L].

LEMMA 17. *Under all the hypothesis of Theorem* 14, *let x_k be a sequence in A which converges to zero in the strong* topology. Then for $\varepsilon > 0$ there is a subsequence x_{k_j} of x_k and a projection p in A such that $\rho(p) > 1 - \varepsilon$ and $\|S_n(x_{k_j})p\| \to 0$ as $j \to \infty$, uniformly in n.*

Proof. Assume first that the x_k are positive. W.L.O.G. assume that $0 \leqslant x_k \leqslant I$. Let $\varepsilon_k = \rho(x_k)$, and note that $\varepsilon_k \to 0$ as x_k converges to zero. Lemma 16 gives a sequence y_k such that $0 < y_k < 4C_{p_1} I$, $\rho(y_k) \leqslant 8(C_{p_1} \varepsilon_k)^{1/2}$ and $S_n(x_k) \leqslant y_k$. Hence, as $\rho(y_k) \to 0$, and ρ is a faithful normal state, then $y_k \to 0$ strongly. The non-commutative Egorov theorem of Saito and Pedersen gives that there is a subsequence y_{k_j} and a projection p such that $\rho(p) > 1 - \varepsilon$ and $\|y_{k_j} p\| \to 0$.

Lemma 5.4 of [L] gives that if $0 \leqslant r \leqslant s \leqslant I$, $r, s \in A$, q a projection such that $\|sq\| < \varepsilon$, then $\|rq\| < \varepsilon^{1/2}$. As $0 \leqslant S_n(x_k) \leqslant y_{k_j} \leqslant I$ then that lemma plus our just acquired result gives that $\|S_n(x_{k_j})p\| \to 0$ as $j \to \infty$, uniformly in n.

Now, suppose x_k is not positive. $x_k = x_{k,1} - x_{k,2} + i(x_{k,3} - x_{k,4})$, where $x_{k,i}$ are positive operators from the standard decomposition of x_k. If x_k converges to zero strong*, then $x_{k,i}$ also converge strong* to zero since $x_{k,i}$ are uniform limits of non-commutative polynomials in x_k and x_k^*. This gives that $z_k = x_{k,1} + x_{k,2} + x_{k,3} + x_{k,4}$ also converges to zero strong*. The first part of this proof gives that there is a projection p such that $\rho(p) > 1 - \varepsilon$ and $\|S_n(z_{k_j})p\| \to 0$ as $j \to \infty$, uniformly in n. As $S_n(x_{k,i}) \leqslant S_n(z_{k_j})$, we have again from 5.4 of [L] that $\|S_n(x_{k_j,i})p\| \to 0$ as $j \to \infty$, uniformly in n. Extending by linearity gives the result. ∎

We remind the reader of the definition of almost uniform convergence. Given A a von Neumann algebra with a faithful normal state ρ, then x_n is said to converge almost uniformly to x if for every $\varepsilon > 0$, there exists a projection $p \in A$ such that $\rho(I - p) < \varepsilon$ and $\|(x_n - x)p\| \to 0$

Theorem 18. *Let $\{\alpha_i\}_{i=1}^r$ be *-automorphisms of a von Neumann algebra A which leave a faithful normal state ρ invariant. Let F_r be the free group on r generators $\{a_i\}_{i=1}^r$, and define the group homomorphism $\phi: F_r \to \mathrm{Aut}(A)$ given by $\phi: a_i \mapsto \alpha_i$. If w_n is the set of all words of F_r of length n, $|w_n|$ the number of elements of w_n, and $\sigma_n = |w_n|^{-1} \sum_{a \in w_n} \phi(a)$. For $x \in A$, $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$ converges almost uniformly to an element $\hat{x}$ in the von Neumann algebra.*

This is an immediate consequence to the stronger result proven, which is

Theorem 19. *Let σ_1 be a normal completely positive map of a von Neumann algebra A, $\sigma_1(1) = 1$, ρ a σ_1-invariant faithful normal state, $p_1 \geqslant p_2 \geqslant 0$, $p_1 + p_2 = 1$, and σ_n $(n \geqslant 2)$ positive maps such that $\sigma_1 \sigma_n = p_1 \sigma_{n+1} + p_2 \sigma_{n-1}$, where $\sigma_0(x) = x$. Let $\Pi(\sigma_1)$ be the action of σ_1 on $L^2(A, \rho)$ given by $\Pi(\sigma_1)\, x\zeta = \pi(\sigma_1(x))\zeta$, where ζ is the cyclic separating vector from the GNS w.r.t ρ, and define $S_n(x) = n^{-1} \sum_{k=0}^{n-1} \sigma_k(x)$. If $\Pi(\sigma_1)$ is a normal operator such that the spectrum $\sigma(\Pi(\sigma_1))$ is contained in D_{p_1}, defined in* (3), *then for $x \in A$, $S_n(x)$ converges almost uniformly to $\hat{x} \in A$.*

Proof. By Theorem 14, we have that there is a bounded sequence x_k in A_u which converges to x strong*. By Lemma 10, $\hat{x}_k \to \hat{x}$ strong*. We replace x by $x - \hat{x}$, x_k by $x_k - \hat{x}_k$ and have that $\hat{x}_k = \hat{x} = 0$. If $z_k = x - x_k$, then z_k is a bounded sequence with strong* limit zero. By Lemma 17, there is a projection p in A, $\rho(p) > 1 - \varepsilon$ such that given any $\delta > 0$ we can find a k for which $\|S_n(z_k)p\| < \frac{1}{2}\delta$ for all n. As x_k are in A_u, and $\hat{x}_k = 0$, there is an integer N such that for all $n > N$, $\|S_n(x_k)\| < \frac{1}{2}\delta$. So, for $n > N$, $\|S_n(x)p\| \leqslant \|S_n(x_k)p\| + \|S_n(z_k)p\| < \delta$, where δ was arbitrary. ∎

APPENDIX A

These are the proofs of the lemmas in Section 2. We provide only rough sketches due to the length required should the proofs be more detailed.

Lemma 20. *Let p be a real number, $0 \leqslant p \leqslant 1$. With $r = \sqrt{4p - 4p^2}$, define functions f and g on the complex plane by $f(z) = (1/4p)(\sqrt{z+r} - \sqrt{z-r})^2$ and $g(z) = (1/4p)(\sqrt{z+r} + \sqrt{z-r})^2$ with $r = \sqrt{4p - 4p^2}$, then, if z_0 is a complex number, the following statements hold.*

1. *If $\frac{1}{2} < p < 1$ then $|f(z_0)| \geqslant 1 \Rightarrow |g(z_0)| < 1$ and $|g(z_0)| \geqslant 1 \Rightarrow |f(z_0)| < 1$.*
2. *If $0 < p < \frac{1}{2}$ then $|f(z_0)| \leqslant 1 \Rightarrow |g(z_0)| > 1$ and $|g(z_0)| \leqslant 1 \Rightarrow |f(z_0)| > 1$.*
3. *If $p = \frac{1}{2}$ then $|f(z_0)| < 1 \Leftrightarrow |g(z_0)| > 1$ and $|f(z_0)| > 1 \Leftrightarrow |g(z_0)| < 1$.*

Proof. Noting that $f(z_0) \neq 0$ (since $p \neq 1$) gives that $f(z_0) = ce^{i\theta}$, $c > 0$. This gives that $g(z_0) = (1-p)/pce^{i\theta}$. The theorem follows from that fact. ∎

Proof of Lemma 1. For brevity, we define $r = \sqrt{4p_1 - 4p_1^2}$. Suppose $z = 1$, then $f_n(z) = p_1 f_{n+1}(z) + p_2 f_{n-1}(z)$ and hence as $f_0(z) = f_1(z) = 1$, then it is clear that $f_n(z)$ is one for all n. So, assume $z \neq 1$.

Suppose $p_1 = 1$. As $f_n(z) = z^n$ then the limit clearly exists only when $|z| < 1$, as the case $z = 1$ is not allowed.

Suppose $p_1 = 0$. Then, the conditions above imply that $z^2 = zf_1(z) = p_2 f_0(z) = 1$ and so $z = \pm 1$. By assumption, $z \neq 1$ and hence $z = -1$. If $z = -1$, then as $-f_n(z) = f_{n-1}(z)$ then $f_n(z) = (-1)^n$ and hence it does not converge.

Hereafter, we assume $0 < p_1 < 1$. It was suggested by Dylan SeLegue that to find a closed form expression for f_n, the standard method which is used to find closed forms for the Fibbonacci numbers and the Bernoulli numbers would be appropriate. In doing so, one can find a closed form. If $z^2 \neq r^2$ then

$$f_n(z) = \frac{1}{2\sqrt{z^2 - r^2}} \left(\left(\frac{(\sqrt{z+r} - \sqrt{z-r})^2}{4p_1} \right)^n (z(1-2p_1) + \sqrt{z^2 - r^2}) \right.$$
$$\left. + \left(\frac{(\sqrt{z+r} + \sqrt{z-r})^2}{4p_1} \right)^n (z(2p_1 - 1) + \sqrt{z^2 - r^2}) \right).$$

If $z^2 = r^2$, then

$$f_n(z) = \pm \left(\sqrt{\frac{1}{p_1}} - 1 \right)^n (1 + n(2p_1 - 1)).$$

We now prove the lemma.

$\Leftarrow$: If $z^2 \neq 4p_1(1-p_1)$ then

$$\left(\frac{(\sqrt{z+r} \pm \sqrt{z-r})^2}{4p_1} \right)^n \to 0$$

and hence $f_n(z)$ goes to zero. If $z^2 = 4p_1(1-p_1)$ then as $p_1 > \frac{1}{2}$, then $\sqrt{(1/p_1) - 1}$ is less than one and hence $f_n(z)$ goes to zero.

$\Rightarrow$: Let $1 > p_1 > \frac{1}{2}$ (i.e. $p_1 > p_2 \geqslant 0$), $z^2 \neq 4p_1(1-p_1)$. We use the notation from Lemma 20. Let $h(z) = (1/4p_1)(\sqrt{z+r} - \sqrt{z-r})^2$ and $g(z) = (1/4p_1)(\sqrt{z+r} + \sqrt{z-r})^2$. From Lemma 20, we know that if $|h(z)| \geqslant 1$ then $(g(z))^n \to 0$, and if $|g(z)| \geqslant 1$ then $(h(z))^n \to 0$. If $|h(z)| > 1$ or $|g(z)| > 1$ then it is clear that z_n cannot converge. Hence, either $|h(z)| = 1$, $|g(z)| = 1$, or both $h(z)$ and $g(z)$ have norm less than one.

If they each have norm less than one, which is exactly when

$$|\sqrt{z+r}+\sqrt{z-r}|<2\sqrt{p_1}$$

and

$$|\sqrt{z+r}-\sqrt{z-r}|<2\sqrt{p_1},$$

then it is clear that $f_n(z)$ converges, and it converges to zero. The rest of the cases are done similarly.

The fact that for some N, if $z \in E_{p_1}$ and $n > N$ then $|f_n(z)| \leqslant 1$ is clear from the definition of E_{p_1} and considering the limit of f_n as $z \to \pm r$. ▮

Proof of Lemma 2. For brevity, we define $r=\sqrt{4p_1-4p_1^2}$. Let h and g be defined as in Lemma 4. If $|h|>1$ or $|g|>1$ then convergence cannot occur as the growth of $f_n(z)$ is exponential from the proof of Lemma 1. Hence, if $f_n(z)$ converges, then $z \in D_{p_1}$. We now show $f_n(z)$ converges for all such z. If $z=\{\pm 1\}$ then the result is obvious. Hence, we may assume that $p_1 \geqslant \frac{1}{2}$. If $z \in E_{p_1}$ then the result that $n^{-1}\sum_{k=0}^{n-1} f_k(z)$ converges to zero is also obvious. Hence, assume $z \in D_{p_1} \backslash \{E_{p_1} \cup \{\pm 1\}\}$.

If $|h| \leqslant 1$ and $|g| \leqslant 1$ then both $n^{-1}\sum_{k=0}^{n-1} g^n$ and $n^{-1}\sum_{k=0}^{n-1} f^n$ converge by either von Neumann's ergodic theorem or by a geometric series argument. Thus, convergence of $n^{-1}\sum_{k=0}^{n-1} f_k(z)$ is shown.

As noted above, convergence to a non-zero number occurs only when $h=1$ or $g=1$. However, this can occur only when $z=1$ in which case the limit is one.

The boundedness of our sums follows from the boundedness in our previous lemma. ▮

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